

A quaternary diophantine inequality by prime numbers of a special type

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Abstract

Let $1 < c < 832/825$. For large real numbers $N > 0$ and a small constant $\vartheta > 0$, the inequality

$$|p_1^c + p_2^c + p_3^c + p_4^c - N| < \vartheta$$

has a solution in prime numbers p_1, p_2, p_3, p_4 such that, for each $i \in \{1, 2, 3, 4\}$, $p_i + 2$ has at most 32 prime factors.

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1 Introduction and statements of the result.

In 1952 I. I. Piatetski-Shapiro [10] investigated the inequality

$$|p_1^c + p_2^c + \cdots + p_r^c - N| < \varepsilon \tag{1}$$

where $c > 1$ is not an integer, ε is a fixed small positive number, and p_1, \dots, p_r are primes. He proved the existence of an $H(c)$, depending only on c , such that for all sufficiently large real N , (1) has a solution for $H(c) \leq r$. He established that

$$\limsup_{c \rightarrow \infty} \frac{H(c)}{c \log c} \leq 4$$

and also that $H(c) \leq 5$ if $1 < c < 3/2$.

In 1992 Tolev [13] proved that (1) has a solution for $r = 3$ and $1 < c < 15/14$. The interval $1 < c < 15/14$ was subsequently improved by several authors [3], [7], [8], [1].

In 2003 Zhai and Cao [15] proved that (1) has a solution for $r = 4$ and $1 < c < 81/68$. Their result was improved to $1 < c < 97/81$ by Mu [9].

In 2016 Dimitrov [4] showed that (1) has a solution for $r = 3$, $0 < c < 4/21$ and primes p_1, p_2, p_3 such that, for each $i \in \{1, 2, 3\}$, $p_i + 2$ has at most 10 prime factors.

Recently Tolev [14] proved that (1) has a solution for $r = 3$, $1 < c < 15/14$ and primes p_1, p_2, p_3 such that, for each $i \in \{1, 2, 3\}$, $p_i + 2$ has at most $\left\lfloor \frac{369}{180-168c} \right\rfloor$ prime factors.

Let P_l is a number with at most l prime factors. Motivated by [14], we shall prove the following theorem.

Theorem 1. *Let A be an arbitrary large and fixed, and let $1 < c < 832/825$. There exists a number $N_0(c) > 0$ such that for each real number $N > N_0(c)$ the inequality*

$$|p_1^c + p_2^c + p_3^c + p_4^c - N| < \frac{1}{(\log N)^A}$$

has a solution in prime numbers p_1, p_2, p_3 such that

$$p_1 + 2 = P'_{32}, \quad p_2 + 2 = P''_{32}, \quad p_3 + 2 = P'''_{32}, \quad p_4 + 2 = P''''_{32}.$$

By choosing the parameters in a different way we may obtain other similar results, for example $1 < c < 51/50$, $p_i + 2 = P_r, i = 1, 2, 3, 4$, where r is large. Obviously the enlargement of the range for c leads to increase of the number of the prime factors of $p_i + 2$.

2 Notations and some lemmas.

As usual $\varphi(n)$ and $\mu(n)$ denote respectively, Euler's function and Möbius' function. We denote by $\tau(n)$ the number of the positive divisors of n . Let (m_1, m_2) be the greatest common divisor. Instead of $m \equiv n \pmod{k}$ we write for simplicity $m \equiv n(k)$. As usual $[y]$ denotes the integer part of y , $e(y) = e^{2\pi i y}$. Let c be a fixed real number such that

$1 < c < 832/825$ and N be a sufficiently large number.

$$X = (N/3)^{1/c}; \quad (2)$$

$$\tau = X^{57/275-c}; \quad (3)$$

$$\vartheta = \frac{1}{(\log X)^{A+1}}, \quad A > 20 \text{ is arbitrary large}; \quad (4)$$

$$K = \frac{\log^2 X}{\vartheta}; \quad (5)$$

$$D = X^{1/11-\varepsilon_0}, \quad \varepsilon_0 = 0.001; \quad (6)$$

$$\eta = \frac{\varepsilon_0}{9}; \quad (7)$$

$$z = X^\beta, \quad 0 < \beta < 1/33; \quad (8)$$

$$P(z) = \prod_{2 < p \leq z} p, \quad p \text{-prime number}; \quad (9)$$

$$I(\alpha) = \int_{X/2}^X e(\alpha t^c) dt. \quad (10)$$

The value of β will be specified latter.

Let $\lambda^\pm(d)$ be the lower and upper bounds Rosser's weights of level D , hence

$$|\lambda^\pm(d)| \leq 1, \quad \lambda^\pm(d) = 0 \quad \text{if} \quad d \geq D \quad \text{or} \quad \mu(d) = 0. \quad (11)$$

For further properties of Rosser's weights we refer to [5], [6].

Lemma 1. *Let $\vartheta \in \mathbb{R}$ and $k \in \mathbb{N}$. There exists a function $\theta(y)$ which is k times continuously differentiable and such that*

$$\begin{aligned} \theta(y) &= 1 & \text{for} & \quad |y| \leq 3\vartheta/4; \\ 0 \leq \theta(y) &< 1 & \text{for} & \quad 3\vartheta/4 < |y| < \vartheta; \\ \theta(y) &= 0 & \text{for} & \quad |y| \geq \vartheta. \end{aligned}$$

and its Fourier transform

$$\Theta(x) = \int_{-\infty}^{\infty} \theta(y) e(-xy) dy$$

satisfies the inequality

$$|\Theta(x)| \leq \min \left(\frac{7\vartheta}{4}, \frac{1}{\pi|x|}, \frac{1}{\pi|x|} \left(\frac{k}{2\pi|x|\vartheta/8} \right)^k \right).$$

Proof. See [11]. □

Lemma 2. *Let $n \in \mathbb{N}$. Then*

$$\tau(n) \ll n^\varepsilon ,$$

where ε is an arbitrary small positive number.

Lemma 3. *Let $X \in \mathbb{R}$, $X \geq 2$. We have*

$$\sum_{n \leq X} \frac{1}{\varphi(n)} \ll \log X .$$

Lemma 4. *Assume that $F(x)$, $G(x)$ are real functions defined in $[a, b]$, $|G(x)| \leq H$ for $a \leq x \leq b$ and $G(x)/F'(x)$ is a monotonous function. Set*

$$I = \int_a^b G(x) e(F(x)) dx .$$

If $F'(x) \geq h > 0$ for all $x \in [a, b]$ or if $F'(x) \leq -h < 0$ for all $x \in [a, b]$ then

$$|I| \ll H/h .$$

If $F''(x) \geq h > 0$ for all $x \in [a, b]$ or if $F''(x) \leq -h < 0$ for all $x \in [a, b]$ then

$$|I| \ll H/\sqrt{h} .$$

Proof. See ([12], p. 71). □

3 Outline of the proof.

Consider the sum

$$\Gamma = \sum_{\substack{X/2 < p_1, p_2, p_3, p_4 \leq X \\ |p_1^c + p_2^c + p_3^c + p_4^c - N| < \vartheta \\ (p_i + 2, P(z)) = 1, i=1, 2, 3, 4}} \log p_1 \log p_2 \log p_3 \log p_4 . \quad (12)$$

Any non-trivial estimate from below of Γ implies the solvability of $|p_1^c + p_2^c + p_3^c + p_4^c - N| < \vartheta$ in primes such that $p_i + 2 = P_h$, $h = [\beta^{-1}]$.

We have

$$\Gamma \geq \tilde{\Gamma} = \sum_{\substack{X/2 < p_1, p_2, p_3, p_4 \leq X \\ (p_i + 2, P(z)) = 1, i=1, 2, 3, 4}} \theta(p_1^c + p_2^c + p_3^c + p_4^c - N) \log p_1 \log p_2 \log p_3 \log p_4 . \quad (13)$$

On the other hand

$$\tilde{\Gamma} = \sum_{X/2 < p_1, p_2, p_3, p_4 \leq X} \theta(p_1^c + p_2^c + p_3^c + p_4^c - N) \Lambda_1 \Lambda_2 \Lambda_3 \Lambda_4 \log p_1 \log p_2 \log p_3 \log p_4, \quad (14)$$

where

$$\Lambda_i = \sum_{d|(p_i+2, P(z))} \mu(d), \quad i = 1, 2, 3, 4.$$

We denote

$$\Lambda_i^\pm = \sum_{d|(p_i+2, P(z))} \lambda^\pm(d), \quad i = 1, 2, 3, 4. \quad (15)$$

From the linear sieve we know that $\Lambda_i^- \leq \Lambda_i \leq \Lambda_i^+$ (see [2], Lemma 10). Then we have a simple inequality

$$\Lambda_1 \Lambda_2 \Lambda_3 \Lambda_4 \geq \Lambda_1^- \Lambda_2^+ \Lambda_3^+ \Lambda_4^+ + \Lambda_1^+ \Lambda_2^- \Lambda_3^+ \Lambda_4^+ + \Lambda_1^+ \Lambda_2^+ \Lambda_3^- \Lambda_4^+ + \Lambda_1^+ \Lambda_2^+ \Lambda_3^+ \Lambda_4^- - 3\Lambda_1^+ \Lambda_2^+ \Lambda_3^+ \Lambda_4^+ \quad (16)$$

(see [2], Lemma 13).

Using (14) and (16) we obtain

$$\begin{aligned} \tilde{\Gamma} \geq \Gamma_0 = & \sum_{X/2 < p_1, p_2, p_3, p_4 \leq X} \theta(p_1^c + p_2^c + p_3^c + p_4^c - N) \\ & \times (\Lambda_1^- \Lambda_2^+ \Lambda_3^+ \Lambda_4^+ + \Lambda_1^+ \Lambda_2^- \Lambda_3^+ \Lambda_4^+ + \Lambda_1^+ \Lambda_2^+ \Lambda_3^- \Lambda_4^+ + \Lambda_1^+ \Lambda_2^+ \Lambda_3^+ \Lambda_4^- - 3\Lambda_1^+ \Lambda_2^+ \Lambda_3^+ \Lambda_4^+) \\ & \times \log p_1 \log p_2 \log p_3 \log p_4. \end{aligned} \quad (17)$$

Let

$$\Gamma_0 = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 - 3\Gamma_5, \quad (18)$$

where for example

$$\Gamma_1 = \sum_{X/2 < p_1, p_2, p_3, p_4 \leq X} \theta(p_1^c + p_2^c + p_3^c + p_4^c - N) \Lambda_1^- \Lambda_2^+ \Lambda_3^+ \Lambda_4^+ \log p_1 \log p_2 \log p_3 \log p_4 \quad (19)$$

and so on.

It is easy to see that $\Gamma_1 = \Gamma_2 = \Gamma_3 = \Gamma_4$. We shall consider the sum Γ_1 . The sum Γ_5 can be considered in the same way.

From (15) and (19) we have

$$\begin{aligned} \Gamma_1 = & \sum_{\substack{d_i | P(z) \\ i=1,2,3,4}} \lambda^-(d_1) \lambda^+(d_2) \lambda^+(d_3) \lambda^+(d_4) \sum_{\substack{X/2 < p_1, p_2, p_3, p_4 \leq X \\ p_i+2 \equiv 0 \pmod{d_i}, i=1,2,3,4}} \theta(p_1^c + p_2^c + p_3^c + p_4^c - N) \\ & \times \log p_1 \log p_2 \log p_3 \log p_4. \end{aligned}$$

Using the inverse Fourier transform for the function $\theta(x)$ we get

$$\begin{aligned}\Gamma_1 &= \sum_{\substack{d_i | P(z) \\ i=1,2,3,4}} \lambda^-(d_1) \lambda^+(d_2) \lambda^+(d_3) \lambda^+(d_4) \sum_{\substack{X/2 < p_1, p_2, p_3, p_4 \leq X \\ p_i + 2 \equiv 0 \pmod{d_i}, i=1,2,3,4}} \log p_1 \log p_2 \log p_3 \log p_4 \\ &\times \int_{-\infty}^{\infty} \Theta(t) e((p_1^c + p_2^c + p_3^c + p_4^c - N)t) dt \\ &= \int_{-\infty}^{\infty} \Theta(t) e(-Nt) L_1(t, X) L_2^3(t, X) dt,\end{aligned}$$

where

$$L_1(t, X) = \sum_{d | P(z)} \lambda^-(d) \sum_{\substack{X/2 < p \leq X \\ p+2 \equiv 0 \pmod{d}}} e(p^c t) \log p, \quad (20)$$

$$L_2(t, X) = \sum_{d | P(z)} \lambda^+(d) \sum_{\substack{X/2 < p \leq X \\ p+2 \equiv 0 \pmod{d}}} e(p^c t) \log p. \quad (21)$$

We divide Γ_1 into three parts

$$\Gamma_1 = \Gamma_1^{(1)} + \Gamma_1^{(2)} + \Gamma_1^{(3)}. \quad (22)$$

where

$$\Gamma_1^{(1)} = \int_{|t| < \tau} \Theta(t) e(-Nt) L_1(t, x) L_2^3(t, X) dt, \quad (23)$$

$$\Gamma_1^{(2)} = \int_{\tau \leq |t| \leq K} \Theta(t) e(-Nt) L_1(t, X) L_2^3(t, X) dt, \quad (24)$$

$$\Gamma_1^{(3)} = \int_{|t| > K} \Theta(t) e(-Nt) L_1(t, X) L_2^3(t, X) dt. \quad (25)$$

We shall estimate $\Gamma_1^{(3)}$, $\Gamma_1^{(1)}$, $\Gamma_1^{(2)}$ respectively in the sections 4, 5, 6. In section 7 we shall complete the proof of the Theorem.

4 Upper bound for $\Gamma_1^{(3)}$.

Arguing as in [14] we obtain

Lemma 5. *For the sum $\Gamma_1^{(3)}$, defined by (25), we have*

$$\Gamma_1^{(3)} \ll 1. \quad (26)$$

5 Asymptotic formula for $\Gamma_1^{(1)}$.

The first lemma we need in this section gives us asymptotic formula for the sums $L_j(\alpha, X)$ denoted by (20) and (21).

Lemma 6. *Let D is defined by (6), and $\lambda(d)$ be complex numbers defined for $d \leq D$ such that*

$$|\lambda(d)| \leq 1, \quad \lambda(d) = 0 \quad \text{if} \quad 2|d \quad \text{or} \quad \mu(d) = 0. \quad (27)$$

If

$$L(\alpha, X) = \sum_{d \leq D} \lambda(d) \sum_{\substack{X/2 < p \leq X \\ p+2 \equiv 0 \pmod{d}}} e(p^c \alpha) \log p$$

then for $|\alpha| < \tau$ we have

$$L(\alpha, X) = I(\alpha) \sum_{d \leq D} \frac{\lambda(d)}{\varphi(d)} + \mathcal{O}\left(\frac{X}{(\log X)^A}\right), \quad (28)$$

where $A > 0$ is an arbitrary large constant.

Proof. See ([14], Lemma 10). □

The next lemma is the following

Lemma 7. *Using the definitions (10), (20) and (21) we have*

$$\begin{aligned} \text{(i)} \quad & \int_{-\tau}^{\tau} |L_j(\alpha, X)|^2 d\alpha \ll X^{2-c} \log^6 X, \quad j = 1, 2 \\ \text{(ii)} \quad & \int_{-\tau}^{\tau} |I(\alpha)|^2 d\alpha \ll X^{2-c} \log X. \end{aligned}$$

Proof. See ([14], Lemma 11). □

Let

$$L_j = L_j(t, X), \quad j = 1, 2$$

$$\mathcal{M}_1 = \mathcal{M}_1(t, X) = I(t) \sum_{d \leq D} \frac{\lambda^-(d)}{\varphi(d)}, \quad (29)$$

$$\mathcal{M}_2 = \mathcal{M}_2(t, X) = I(t) \sum_{d \leq D} \frac{\lambda^+(d)}{\varphi(d)}. \quad (30)$$

where $L_j(t, X)$ are denoted by (20) and (21).

We use the identity

$$\begin{aligned} L_1 L_2^3 &= \mathcal{M}_1 \mathcal{M}_2^3 + (L_1 - \mathcal{M}_1) \mathcal{M}_2^3 + L_1 (L_2 - \mathcal{M}_2) \mathcal{M}_2^2 \\ &\quad + L_1 L_2 (L_2 - \mathcal{M}_2) \mathcal{M}_2 + L_1 L_2^2 (L_2 - \mathcal{M}_2). \end{aligned} \quad (31)$$

Replace

$$J_1 = \int_{|t| < \tau} \Theta(t) e(-Nt) \mathcal{M}_1(t, X) \mathcal{M}_2^3(t, X) dt. \quad (32)$$

Then from Lemma 1, Lemma 6, (20), (21), (23), (29) – (32) we obtain

$$\begin{aligned} \Gamma_1^{(1)} - J_1 &= \int_{|t| < \tau} \Theta(t) e(\eta t) \left(L_1(t, X) - \mathcal{M}_1(t, X) \right) \mathcal{M}_2^3(t, X) dt \\ &\quad + \int_{|t| < \tau} \Theta(t) e(\eta t) L_1(t, X) \left(L_2(t, X) - \mathcal{M}_2(t, X) \right) \mathcal{M}_2^2(t, X) dt \\ &\quad + \int_{|t| < \tau} \Theta(t) e(\eta t) L_1(t, X) L_2(t, X) \left(L_2(t, X) - \mathcal{M}_2(t, X) \right) \mathcal{M}_2(t, X) dt \\ &\quad + \int_{|t| < \tau} \Theta(t) e(\eta t) L_1(t, X) L_2^2(t, X) \left(L_2(t, X) - \mathcal{M}_2(t, X) \right) dt \\ &\ll \vartheta \frac{X}{(\log X)^A} \left(\int_{|t| < \tau} |\mathcal{M}_2^3(t, X)| dt + \int_{|t| < \tau} |L_1(t, X) \mathcal{M}_2^2(t, X)| dt \right. \\ &\quad \left. + \int_{|t| < \tau} |L_1(t, X) L_2(t, X) \mathcal{M}_2(t, X)| dt + \int_{|t| < \tau} |L_1(t, X) L_2^2(t, X)| dt \right). \end{aligned} \quad (33)$$

On the other hand (11), (30) and Lemma 3 give us

$$|\mathcal{M}_2(t, X)| \ll |I(t)| \log X. \quad (34)$$

Using (33) and (34) we find

$$\begin{aligned} \Gamma_1^{(1)} - J_1 &\ll \vartheta \frac{X}{(\log X)^{A-3}} \left(\int_{|t| < \tau} |I(t)|^3 dt + \int_{|t| < \tau} |L_1(t, X)| |I(t)|^2 dt \right. \\ &\quad \left. + \int_{|t| < \tau} |L_1(t, X) L_2(t, X) I(t)| dt + \int_{|t| < \tau} |L_1(t, X) L_2^2(t, X)| dt \right). \end{aligned} \quad (35)$$

Bearing in mind the definitions (10), (20) and (21) we get the trivial estimates

$$|I(t)| \ll X; \quad |L_j(t, X)| \ll X \log^2 X, \quad j = 1, 2. \quad (36)$$

Now from (35), (36) and Lemma 7 we obtain

$$\Gamma_1^{(1)} - J_1 \ll \vartheta \frac{X^2}{(\log X)^{A-5}} \left(\int_{|t| < \tau} |I(t)|^2 dt + \int_{|t| < \tau} |L_1(t, X)|^2 dt \right) \ll \vartheta \frac{X^{4-c}}{(\log X)^{A-11}}. \quad (37)$$

Let us consider J_1 . According to Lemma 4 we have

$$|I(\alpha)| \ll \frac{X^{1-c}}{|\alpha|}. \quad (38)$$

Therefore by Lemma 1, Lemma 3, (29), (30), (32) and (38) we find

$$\begin{aligned} J_1 &= \sum_{\substack{d_i | P(z) \\ i=1,2,3,4}} \frac{\lambda^-(d_1) \lambda^+(d_2) \lambda^+(d_3) \lambda^+(d_4)}{\varphi(d_1) \varphi(d_2) \varphi(d_3) \varphi(d_4)} \int_{|t| < \tau} \Theta(t) e(-Nt) \\ &\quad \times \left(\int_{X/2}^X e(ty_1^c) dy_1 \int_{X/2}^X e(ty_2^c) dy_2 \int_{X/2}^X e(ty_3^c) dy_3 \int_{X/2}^X e(ty_4^c) dy_4 \right) dt \\ &= \sum_{\substack{d_i | P(z) \\ i=1,2,3,4}} \frac{\lambda^-(d_1) \lambda^+(d_2) \lambda^+(d_3) \lambda^+(d_4)}{\varphi(d_1) \varphi(d_2) \varphi(d_3) \varphi(d_4)} \left[\int_{-\infty}^{\infty} \Theta(t) e(-Nt) \right. \\ &\quad \times \left(\int_{X/2}^X \int_{X/2}^X \int_{X/2}^X \int_{X/2}^X e(t(y_1^c + y_2^c + y_3^c + y_4^c)) dy_1 dy_2 dy_3 dy_4 \right) dt \\ &\quad \left. + \mathcal{O} \left(\vartheta X^{4-4c} \int_{\tau}^{\infty} \frac{dt}{t^4} \right) \right] \\ &= \sum_{\substack{d_i | P(z) \\ i=1,2,3,4}} \frac{\lambda^-(d_1) \lambda^+(d_2) \lambda^+(d_3) \lambda^+(d_4)}{\varphi(d_1) \varphi(d_2) \varphi(d_3) \varphi(d_4)} \left(\int_{X/2}^X \int_{X/2}^X \int_{X/2}^X \int_{X/2}^X \right. \\ &\quad \times \int_{-\infty}^{\infty} \Theta(t) e(t(y_1^c + y_2^c + y_3^c + y_4^c - N)) dt dy_1 dy_2 dy_3 dy_4 + \mathcal{O}(\vartheta X^{4-4c} \tau^{-3}) \Big) \end{aligned}$$

$$\begin{aligned}
&= \sum_{\substack{d_i | P(z) \\ i=1,2,3,4}} \frac{\lambda^-(d_1)\lambda^+(d_2)\lambda^+(d_3)\lambda^+(d_4)}{\varphi(d_1)\varphi(d_2)\varphi(d_3)\varphi(d_4)} \\
&\times \left(\int_{X/2}^X \int_{X/2}^X \int_{X/2}^X \int_{X/2}^X \theta(y_1^c + y_2^c + y_3^c + y_4^c - N) dy_1 dy_2 dy_3 dy_4 + \mathcal{O}(\vartheta X^{4-4c} \tau^{-3}) \right) \\
&= \int_{X/2}^X \int_{X/2}^X \int_{X/2}^X \int_{X/2}^X \theta(y_1^c + y_2^c + y_3^c + y_4^c - N) dy_1 dy_2 dy_3 dy_4 \\
&\times \sum_{\substack{d_i | P(z) \\ i=1,2,3,4}} \frac{\lambda^-(d_1)\lambda^+(d_2)\lambda^+(d_3)\lambda^+(d_4)}{\varphi(d_1)\varphi(d_2)\varphi(d_3)\varphi(d_4)} + \mathcal{O}(\vartheta X^{4-4c} \tau^{-3} \log^4 X) .
\end{aligned}$$

The last formula, (3) and (37) imply

$$\Gamma_1^{(1)} = B(X) \sum_{d|P(z)} \frac{\lambda^-(d)}{\varphi(d)} \left(\sum_{d|P(z)} \frac{\lambda^+(d)}{\varphi(d)} \right)^3 + \mathcal{O} \left(\vartheta \frac{X^{4-c}}{(\log X)^{A-11}} \right) , \quad (39)$$

where

$$B(X) = \int_{X/2}^X \int_{X/2}^X \int_{X/2}^X \int_{X/2}^X \theta(y_1^c + y_2^c + y_3^c + y_4^c - N) dy_1 dy_2 dy_3 dy_4 . \quad (40)$$

According to ([15], Lemma 8) we have

$$B(X) \gg \vartheta X^{4-c} . \quad (41)$$

Let

$$G^\pm = \sum_{d|P(z)} \frac{\lambda^\pm(d)}{\varphi(d)} . \quad (42)$$

Thus from (39) and (42) it follows

$$\Gamma_1^{(1)} = B(X) G^- (G^+)^3 + \mathcal{O} \left(\vartheta \frac{X^{4-c}}{(\log X)^{A-11}} \right) . \quad (43)$$

6 Upper bound for $\Gamma_1^{(2)}$.

The treatment of the intermediate region depends on the following four lemmas.

Lemma 8. For the sums denoted by (20) and (21) we have

$$\int_0^1 |L_j(t, X)|^2 dt \ll X \log^5 X, \quad j = 1, 2.$$

Proof. See ([14], Lemma 11). □

Lemma 9. Let $1 < c < 1603/1033$. Then

$$\sum_{X/2 < n_1, n_2, n_3, n_4 \leq X} \min \left(1, \frac{1}{|n_1^c + n_2^c - n_3^c - n_4^c|} \right) \ll X^{4-c} \log^5 X.$$

Proof. See ([15], Theorem 1). □

Lemma 10. For the sums denoted by (20) and (21) we have

$$\int_0^1 |L_j(t, X)|^4 dt \ll X^{4-c+\eta}, \quad j = 1, 2,$$

where η is defined by (7).

Proof. We only prove for $j = 1$. The case for $j = 2$ is analogous.

From (11), (20), Lemma 2 and Lemma 9 it follows

$$\begin{aligned} & \int_0^1 |L_1(t, X)|^4 dt = \\ &= \sum_{\substack{d_i | P(z) \\ i=1,2,3,4}} \lambda^-(d_1) \cdots \lambda^-(d_4) \sum_{\substack{X/2 < p_1, p_2, p_3, p_4 \leq X \\ p_i + 2 \equiv 0 \pmod{d_i}, i=1,2,3,4}} \log p_1 \cdots \log p_4 \int_0^1 e((p_1^c + p_2^c - p_3^c - p_4^c)t) dt \\ &\ll \sum_{\substack{d_i \leq D \\ i=1,2,3,4}} \sum_{\substack{X/2 < p_1, p_2, p_3, p_4 \leq X \\ p_i + 2 \equiv 0 \pmod{d_i}, i=1,2,3,4}} \log p_1 \cdots \log p_4 \min \left(1, \frac{1}{|p_1^c + p_2^c - p_3^c - p_4^c|} \right) \\ &\ll (\log X)^4 \sum_{\substack{d_i \leq D \\ i=1,2,3,4}} \sum_{\substack{X/2 < n_1, n_2, n_3, n_4 \leq X \\ n_i + 2 \equiv 0 \pmod{d_i}, i=1,2,3,4}} \min \left(1, \frac{1}{|n_1^c + n_2^c - n_3^c - n_4^c|} \right) \\ &= (\log X)^4 \sum_{X/2 < n_1, n_2, n_3, n_4 \leq X} \min \left(1, \frac{1}{|n_1^c + n_2^c - n_3^c - n_4^c|} \right) \sum_{\substack{d_1 \leq D \\ d_1 | n_1 + 2}} 1 \cdots \sum_{\substack{d_4 \leq D \\ d_4 | n_4 + 2}} 1 \\ &\ll (\log X)^4 \sum_{X/2 < n_1, n_2, n_3, n_4 \leq X} \min \left(1, \frac{1}{|n_1^c + n_2^c - n_3^c - n_4^c|} \right) \tau(n_1 + 2) \cdots \tau(n_4 + 2) \end{aligned}$$

$$\begin{aligned} &\ll X^{\eta/2} \sum_{X/2 < n_1, n_2, n_3, n_4 \leq X} \min \left(1, \frac{1}{|n_1^c + n_2^c - n_3^c - n_4^c|} \right) \\ &\ll X^{4-c+\eta}. \end{aligned}$$

□

Lemma 11. Assume that $\tau \leq |\alpha| \leq K$. Let $\beta(d)$ be complex number defined for $d \leq D$, and let

$$\beta(d) \ll 1. \quad (44)$$

Then for the sum

$$L(\alpha, X) = \sum_{d \leq D} \beta(d) \sum_{\substack{X/2 < p \leq X \\ p+2 \equiv 0 \pmod{d}}} e(p^c \alpha) \log p \quad (45)$$

we have

$$L(\alpha, X) \ll X^\eta \left(X^{1/3+c/2} D K^{1/2} + X^{3/4+c/6} D^{2/3} K^{1/6} + X^{1-c/6} D^{1/3} \tau^{-1/6} \right),$$

where η is defined by (7).

Proof. See ([14], Lemma 15). □

We next treat $\Gamma_1^{(2)}$, defined by (24). We have

$$\Gamma_1^{(2)} \ll \max_{\tau \leq t \leq K} |L_1(t, X)| \int_{\tau}^K |\Theta(t)| |L_2(t, X)|^3 dt. \quad (46)$$

Using Cauchy's inequality we obtain

$$\int_{\tau}^K |\Theta(t)| |L_2(t, X)|^3 dt \ll \left(\int_{\tau}^K |\Theta(t)| |L_2(t, X)|^2 dt \right)^{1/2} \left(\int_{\tau}^K |\Theta(t)| |L_2(t, X)|^4 dt \right)^{1/2}. \quad (47)$$

On the one hand from (4), (5), Lemma 1 and Lemma 8 it follows

$$\begin{aligned} \int_{\tau}^K |\Theta(t)| |L_2(t, X)|^2 dt &\ll \vartheta \int_{\tau}^{1/\vartheta} |L_2(t, X)|^2 dt + \int_{1/\vartheta}^K |L_2(t, X)|^2 \frac{dt}{t} \\ &\ll \vartheta \sum_{0 \leq n \leq 1/\vartheta} \int_n^{n+1} |L_2(t, X)|^2 dt + \sum_{1/\vartheta-1 \leq n \leq K} \frac{1}{n} \int_n^{n+1} |L_2(t, X)|^2 dt \\ &\ll X \log^6 X. \end{aligned} \quad (48)$$

On the other hand (4), (5), Lemma 1 and Lemma 10 give us

$$\begin{aligned}
\int_{\tau}^K |\Theta(t)| |L_2(t, X)|^4 dt &\ll \vartheta \int_{\tau}^{1/\vartheta} |L_2(t, X)|^4 dt + \int_{1/\vartheta}^K |L_2(t, X)|^4 \frac{dt}{t} \\
&\ll \vartheta \sum_{0 \leq n \leq 1/\vartheta} \int_n^{n+1} |L_2(t, X)|^4 dt + \sum_{1/\vartheta - 1 \leq n \leq K} \frac{1}{n} \int_n^{n+1} |L_2(t, X)|^4 dt \\
&\ll X^{4-c+\eta} \log X,
\end{aligned} \tag{49}$$

where η is defined by (7).

Therefore by (3) – (7), (46) – (49) and by Lemma 11 we obtain

$$\Gamma_1^{(2)} \ll \vartheta \frac{X^{4-c}}{\log^5 X}. \tag{50}$$

Summarizing (22), (26), (43) and (50) we find

$$\Gamma_1 = B(X)G^-(G^+)^3 + \mathcal{O}\left(\vartheta \frac{X^{4-c}}{\log^5 X}\right). \tag{51}$$

7 Proof of the Theorem.

Since $\Gamma_1 = \Gamma_2 = \Gamma_3 = \Gamma_4$ and Γ_5 is estimated in the same way then from (12), (13), (17), (18) and (51) we obtain

$$\Gamma \geq B(X)W + \mathcal{O}\left(\vartheta \frac{X^{4-c}}{\log^5 X}\right), \tag{52}$$

where

$$W = 4(G^+)^3 \left(G^- - \frac{3}{4}G^+\right). \tag{53}$$

We put

$$\mathcal{F}(z) = \prod_{2 < p \leq z} \left(1 - \frac{1}{p-1}\right), \quad s = \frac{\log D}{\log z}. \tag{54}$$

Let $f(s)$ and $F(s)$ are the lower and the upper functions of the linear sieve. Using (42) and ([2], Lemma 10) we obtain

$$\begin{aligned}
&\mathcal{F}(z) \left(f(s) + \mathcal{O}((\log X)^{-1/3}) \right) \\
&\leq G^- \leq \mathcal{F}(z) \leq G^+ \\
&\leq \mathcal{F}(z) \left(F(s) + \mathcal{O}((\log X)^{-1/3}) \right).
\end{aligned} \tag{55}$$

To estimate W from below we shall use the inequalities (see (55)):

$$\begin{aligned} G^- - \frac{2}{3}G^+ &\geq \mathcal{F}(z) \left(f(s) - \frac{3}{4}F(s) + \mathcal{O}((\log X)^{-1/3}) \right) \\ G^+ &\geq \mathcal{F}(z). \end{aligned} \quad (56)$$

Then from (53) and (56) it follows

$$W \geq 4\mathcal{F}^4(z) \left(f(s) - \frac{3}{4}F(s) + \mathcal{O}((\log X)^{-1/3}) \right). \quad (57)$$

Hence, using (52) and (57) we get

$$\Gamma \geq 4B\mathcal{F}^4(z) \left(f(s) - \frac{2}{3}F(s) + \mathcal{O}((\log X)^{-1/3}) \right) + \mathcal{O}\left(\vartheta \frac{X^{4-c}}{\log^5 X}\right). \quad (58)$$

For $2 \leq s \leq 3$ we have

$$f(s) = \frac{2e^\gamma \log(s-1)}{s}, \quad F(s) = \frac{2e^\gamma}{s}$$

(γ denotes Euler's constant). We choose

$$s = 2.95.$$

Then by (8), (6) and (54) we find

$$\beta = 0.030477.$$

It is not difficult to compute that for sufficiently large X we have

$$f(s) - \frac{2}{3}F(s) > 10^{-5}. \quad (59)$$

It remains to notice that

$$\mathcal{F}(z) \asymp \frac{1}{\log X}. \quad (60)$$

Therefore, using (8), (41), (58) – (60) we obtain

$$\Gamma \gg \vartheta \frac{X^{4-c}}{\log^4 X}. \quad (61)$$

From (4) and (61) it follows that $\Gamma \rightarrow \infty$ as $X \rightarrow \infty$.

Bearing in mind (4), (12) and (61) we conclude that for some constant $c_0 > 0$ there are at least $c_0 X^{4-c} \log^{-A-9} X$ triples of primes p_1, p_2, p_3 satisfying $X/2 < p_1, p_2, p_3, p_4 \leq X$, $|p_1^c + p_2^c + p_3^c + p_4^c - N| < \vartheta$ and such that for every prime factor p of $p_j + 2$, $j = 1, 2, 3, 4$ we have $p \geq X^{0.030477}$.

The proof of the Theorem is complete.

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